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Interpolating Sequences for Besov Spaces

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We characterise the interpolating sequences for the Besov spaces B_p and for their multiplier spaces. We also construct linear operators of interpolation.

Key Words: interpolation; Besov spaces.

1. INTRODUCTION

We consider interpolating sequences for holomorphic Besov spaces, introduced by the Möbius invariant semi-norm

$$\|f\|_{B_p}^p = \int_U (|f'(z)|(1-|z|^2))^p(1-|z|^2)^{-2}.$$

These are the sequences $\{z_n\}$ for which the map $f \mapsto \frac{f(z_n)}{a_n}$ transforms B_p onto and into l^p, where a_n is the norm of the point-evaluation functional at z_n. These sequences can be seen to be exactly those for which we have free interpolation for B_p. This means that whenever $\{w_n\}$ satisfies $|w_n| \leqslant |f(z_n)|$ for some f in B_p, we can find g in B_p with $g(z_n) = w_n$.

The multipliers of B_p are those φ for which $\varphi \cdot f$ is in B_p whenever f is in B_p. The corresponding notion of interpolating sequences for such functions is that the map $f \mapsto f(z_n)$ should transform the multipliers of B_p onto l^∞.

The problem of characterising the interpolating sequences for multipliers of B_2 was posed in [2], which also contained an existence result. This was taken up in [7], where a complete description was given. It was also proved in [7] that for $p = 2$ these two notions of interpolation coincide in that they are described by the same sequences.

The purpose of the present paper is to give a similar characterisation for B_p, where $1 < p < \infty$. In [7] Hilbert space techniques are used extensively. In particular, the interpolating functions are not constructed, but only shown to exist. The main point in our approach is the use of a reproducing-type formula which allows us to construct an interpolating analytic function. The current proof is also, in our opinion, simpler for the case $p = 2$ than the one

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in [7]. We also construct bounded linear extension operators for these interpolation problems, that is, inverses to the restriction maps.

As for bounded analytic functions (see [3]), the characterisation of the interpolating sequences involves a separation condition that tells how close two points in the sequence can be. This is expressed in the hyperbolic metric

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}\ ,$$

where ρ is the pseudohyperbolic distance

$$\rho(z,w) = |\phi_z(w)| = |\frac{z-w}{1-\bar{z}w}|.$$

Analogously as for H^p, we define the Carleson measures for B_p by requiring that the inclusion into L^p is continuous, that is,

$$\int_U |f(z)|^p\, d\mu(z) \leqslant C\cdot \|f\|^p_{B_p}.$$

The role of these measures in the characterisation of interpolating sequences is similar to the H^p situation.

It will be convenient for us to identify a multiplier φ with the operator given by $M_\varphi(f) = \varphi \cdot f$. By using the operator norm as the multiplier norm, we introduce the Banach space M_{B_p} of multipliers.

Throughout we make the convention $f(0) = 0$, so that $\|\cdot\|_{B_p}$ is a norm. We will make extensive use of the pairing

$$\langle f, g\rangle = \int_U f'\overline{g'},$$

where integration is against Lebesgue measure normalised to have mass 1. When otherwise is not stated, integration will be against this measure. The pairing above gives a duality $B_p^* = B_q, \frac{1}{p}+\frac{1}{q} = 1$. Here, we must remark that the B_q norm is equivalent to the functional norm (see [18, 5.3.7] for this). There are also reproducing kernels k_w in the sense

$$f(w) = \langle f, k_w\rangle,$$

which by a simple computation (see e.g. [7]) can be seen to be given explicitly by $k_w(z) = \log\frac{1}{1-\bar{w}z}$.

We can now state our main theorem.

THEOREM 1.1. *The following conditions are equivalent*:

$$\text{The map } f \mapsto \{f(z_n)\} \text{ from } M_{B_p} \text{ is bounded and onto } l^\infty, \tag{1}$$

$$\left\|\sum b_n k_{z_n}\right\|_{B_q} \leqslant C \cdot \left\|\sum a_n k_{z_n}\right\|_{B_q} \text{ whenever } |b_n| \leqslant |a_n|, \tag{2}$$

$$\left\|\sum a_n \frac{k_{z_n}}{\|k_{z_n}\|_{B_q}}\right\|_{B_q} \sim \|a_n\|_{l^q}, \tag{3}$$

$$\text{The map } f \mapsto \left\{\frac{f(z_n)}{\|k_{z_n}\|_{B_q}}\right\} \text{ from } B_p \text{ is bounded and onto } l^p, \tag{4}$$

$$\beta(z_n, z_m) \geqslant C \cdot \beta(z_n, 0) \text{ and } \sum \|k_{z_n}\|_{B_q}^{-p} \delta_{z_n} \text{ is a Carleson measure for } B_p. \tag{5}$$

We point out that in the case $p = 2$, Theorem 1.1 is proved in [7]. Condition (2) states that the reproducing kernels corresponding to $\{z_n\}$ are an unconditional basic sequence in the dual space. A similar formulation holds for interpolation in H^p spaces (see [9, p. 188]). We should also mention that a general discussion of unconditional bases, together with a number of conditions equivalent to (2), can be found in [13, 17.1]. The interpolations in (1) and (4) can be done linearly, as stated in the next two theorems.

THEOREM 1.2. *Suppose* (5) *holds, then we can find* h_{z_n} *satisfying*

$$h_{z_n}(z_m) = \delta_{n,m} \qquad \text{and}$$

$$\text{The map } \{w_n\} \mapsto \sum_n w_n h_{z_n} \text{ is bounded from } l^\infty \text{ to } M_{B_p}.$$

THEOREM 1.3. *Suppose* (5) *holds, then we can find* h_{z_n} *satisfying*

$$h_{z_n}(z_m) = \delta_{n,m} \qquad \text{and}$$

$$\text{The map } \left\{\frac{w_n}{\|k_{z_n}\|_{B_q}}\right\} \mapsto \sum_n w_n h_{z_n} \text{ is bounded from } l^p \text{ to } B_p.$$

The proofs of these theorems give explicit constructions of such functions. The spaces we denote by B_p are part of a larger scale of Besov spaces, where they are denoted by $B_{pp}^{1/p}$. The analogous interpolation problems for the

Besov spaces B^s_{pp} with $sp<1$ have been considered in [4, 17]. However, in these cases, the methods used are adaptations of those developed for H^p and for bounded analytic functions. For B_p these techniques do not readily apply, mainly because there are no Blaschke products in these spaces. In this connection we should also mention the papers [8, 12] where similar interpolation problems for other function spaces are considered. Finally, we remark that the recent preprint [10] contains alternative characterisations of Carleson measures for B_p as well as some partial results in the direction of Theorem 1.1.

2. PRELIMINARIES ON BESOV SPACES AND THE MULTIPLIER SPACES

In the description of Carleson measures, we will use the capacity associated with the function space. To define this, we need the space B^r_p of boundary values of the real parts of functions in B_p. It is normed by

$$\|g\|_{B^r_p} = \|g\|_{L_p(T)} + \left(\int_0^{2\pi}\int_0^{2\pi} \frac{|g(e^{i(\theta+t)}) - g(e^{i\theta})|^p}{t^2}\, d\theta\, dt\right)^{1/p}.$$

The capacity is defined for $K \subset T$ by

$$C_p(K) = \inf \|g\|^p_{B^r_p},$$

where infimum is taken over those g with $g \geqslant 1$ on K.

The Carleson measures are characterised explicitly by the following capacitary condition:

$$\mu\left(\bigcup_i S(I_i)\right) \leqslant C \cdot C_p\left(\bigcup_i I_i\right), \tag{6}$$

where I_i is a collection of disjoint intervals and $S(I_i)$ are associated Carleson boxes defined by $S(I) = \{z\colon \frac{z}{|z|} \in I,\ 1 - |I| \leqslant |z| \leqslant 1\}$. We will also need the observation that the p'th root of the best constant in (6) is comparable to the Carleson norm, by which we mean the norm of the imbedding operator. References for this material are [6, 16].

Another useful fact is that the C_p capacity of an interval is comparable to $\log(\frac{1}{|I|})^{-(p-1)}$, where $|I|$ is the length of the interval. This can be found e.g. in [6].

The connection between Carleson measures and multipliers is the following.

THEOREM 2.1. *f is a multiplier of B_p if and only if $|f'|^p(1-|z|)^{p-2}$ is a Carleson measure for B_p and f is bounded. Further, the multiplier norm is bounded by the Carleson norm plus the H^∞ – norm.*

A proof of this is in [16]. Capacity has a property similar to that of a doubling measure. This is surely not unexpected, and we now give a simple proof.

LEMMA 2.1. *Suppose I_i is a collection of disjoint intervals, then we have*

$$C_p\left(\bigcup_i 2I_i\right) \leqslant C \cdot C_p\left(\bigcup_i I_i\right).$$

Proof. Choose g in B_p^r so that $g \geqslant 1$ on $\bigcup_i I_i$. Then $Mg \geqslant \frac{1}{4}$ on $\bigcup_i 2I_i$ where M is the maximal operator. The statement follows by the boundedness of the maximal operator on B_p^r (see e.g. [16, p. 153] for an argument). ∎

We now state a computational lemma.

LEMMA 2.2. *The integral*

$$\int_U \frac{(1-|\zeta|)^t}{|1-\bar{\zeta}z|^{2+t+c}}, \qquad t > -1$$

is comparable to $(1-|z|)^{-c}$ when $c > 0$ and to $\log\left(\frac{1}{1-|z|}\right)$ when $c = 0$.

A proof is in e.g. [18, 4.2.2]. Using this lemma, we can be more explicit about the norms of the reproducing kernels that appear in (5), that is,

$$||k_w||_{B_q}^q \sim \int_U \frac{(1-|z|)^{q-2}}{|1-\bar{w}z|^q} \sim \log\left(\frac{1}{1-|w|}\right) \sim \beta(w,0). \tag{7}$$

The separation condition in (5) is connected to the regularity of functions in B_p inside the disk. This is described by a Hölder-type estimate in the hyperbolic metric.

LEMMA 2.3. *For $f \in B_p$, $\frac{1}{p}+\frac{1}{q}=1$, we have the estimate*

$$|f(z)-f(w)| \leqslant C_p \cdot ||f||_{B_p} \cdot \beta(z,w)^{1/q}.$$

For proof, see [19, Theorem D], or note that it follows from the case $w=0$ by the Möbius invariance of both the hyperbolic metric and the B_p semi-norm, and that this special case is contained in estimate (7) of the reproducing kernel.

Our approach for constructing the interpolating multipliers is related to the following reproducing formula for analytic functions (see e.g. [16, p. 162]):

$$f(z) = f(0) + C_s \int_U \frac{f'(\zeta)(1-|\zeta|^2)^s}{\bar{\zeta}(1-\bar{\zeta}z)^{1+s}}. \tag{8}$$

More precisely, we will construct analytic functions of the form

$$f(z) = \int_U \frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{1+s}}, \tag{9}$$

where g is non-analytic. We will need a characterisation of multipliers suited to this representation. Observe that for functions f given by (9), we have $f' = C_s \cdot T_s(g)$, where T_s is the operator given by

$$T_s(g) = \int \frac{\bar{\zeta}g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{2+s}}.$$

The following lemma will give us what we need.

Lemma 2.4. *Suppose $|g|^p(1-|z|)^{p-2}$ is a Carleson measure for B_p, then for $s > \frac{1}{p}$, $|T_s g|^p(1-|z|)^{p-2}$ is also a Carleson measure of norm bounded by that for $|g|^p(1-|z|)^{p-2}$ plus $\|g(z)(1-|z|)\|_{L^\infty}$.*

Proof. First, we remark that T_s is bounded on $L^p((1-|z|)^{p-2})$ whenever $s > -\frac{1}{p}$ (see [20]). Now choose any h in B_p. Since $h \cdot g$ has $L^p((1-|z|)^{p-2})$ norm bounded by $C \cdot \|h\|_{B_p}$, where C is the Carleson norm of $|g|^p(1-|z|)^{p-2}$, we only need an estimate on $hT_s g - T_s(hg)$. Pointwise, we have

$$h(z)T_s g(z) - T_s(hg)(z) = \int \bar{\zeta}g(\zeta)\frac{h(z)-h(\zeta)}{(1-\bar{\zeta}z)^{2+s}}(1-|\zeta|)^s.$$

We choose q so that $\frac{1}{q} + \frac{1}{p} = 1$. Hölder's inequality then gives that the above is bounded by

$$\|g(z)(1-|z|)\|_{L^\infty} \cdot \left(\int \frac{|h(z)-h(\zeta)|^p}{|1-\bar{\zeta}z|^4}\right)^{1/p} \cdot \left(\int \frac{(1-|\zeta|)^{(s-1)\cdot q}}{|1-\bar{\zeta}z|^{(2+s-4/p)\cdot q}}\right)^{1/q}.$$

By choosing s large enough for the last factor to be finite ($s > \frac{1}{p}$), we see from Lemma 2.2 that it is comparable to $(1-|z|)^{(2-p)/p}$. The $L^p((1-|z|)^{p-2})$-norm of $hT_s g - T_s(hg)$ is hence bounded by

$$\|g(z)(1-|z|)\|_{L^\infty} \cdot \left(\int \frac{|h(z)-h(\zeta)|^p}{|1-\bar{\zeta}z|^4}\, dA(z)\, dA(\zeta)\right)^{1/p}.$$

The second factor is comparable to the B_p norm of h, by the description of B_p in [18, 5.3.4]. This gives the desired estimate. ∎

Some remarks on this lemma and its proof seem to be in order.

(i) The main point in the calculation above is clearly that $hT_sg - T_s \times (hg)$ is essentially insignificant compared to $T_s(hg)$.

(ii) It would be tempting to prove this lemma using the capacitary description of Carleson measures, but this condition seems difficult to check since it involves collections of intervals.

3. PROOF OF THE NECESSITY OF CONDITION (5)

(1) implies (2): We choose φ so that $\overline{\varphi(z_n)}a_n = b_n$. As in [7], a reproducing kernel is an eigenvector of the adjoint of the multiplication operator. More precisely, $M_\varphi^*(k_{z_n}) = \overline{\varphi(z_n)}k_{z_n}$ holds. This gives

$$\begin{aligned}\left\|\sum b_n k_{z_n}\right\|_{B_q} &= \left\|M_\varphi^*\left(\sum a_n k_{z_n}\right)\right\|_{B_q}\\ &\leqslant \|M_\varphi\| \cdot \left\|\sum a_n k_{z_n}\right\|_{B_q} \leqslant C \cdot \|\varphi(z_n)\|_{l^\infty} \cdot \left\|\sum a_n k_{z_n}\right\|_{B_q},\end{aligned}$$

where the last inequality follows from the open mapping theorem.

(2) implies (3): Our proof will be based on a rather curious lemma. We will formulate it in some generality, since we believe it could be of independent interest.

Lemma 3.1. *Suppose f_n is an unconditional basic sequence in $L^p(d\mu)$ of positive functions. Then it holds*

$$\| \, \|a_n f_n\|_{l^1} \|_{L^p} \leqslant C \cdot \| \, \|a_n f_n\|_{l^\infty} \|_{L^p}.$$

Proof. We will make use of the Rademacher functions r_n and the Khintchine inequality stating that for functions in the linear span of r_n, all L^p norms are comparable (see e.g. [14, IV, 5.2]). This inequality will reduce an L^p norm to an L^2 norm, allowing us to use the orthogonality of the Rademacher functions. The calculation is as follows:

$$\begin{aligned}\left\|\sum a_n f_n\right\|_{L^p}^p &\leqslant C \cdot \int_0^1 \left\|\sum r_n(t) a_n f_n\right\|_{L^p}^p dt\\ &= C \cdot \int \int_0^1 \left|\sum r_n(t) a_n f_n\right|^p dt\, d\mu \leqslant C \cdot \int \|a_n f_n\|_{l^2}^p \, d\mu.\end{aligned}$$

Now, since we have

$$\|a_n f_n\|_{l^2}^2 \leqslant \|a_n f_n\|_{l^1} \cdot \|a_n f_n\|_{l^\infty},$$

an application of Schwarz inequality gives

$$\int \|a_n f_n\|_{l^2}^p \, d\mu \leqslant \left(\int \|a_n f_n\|_{l^1}^p \, d\mu \right)^{1/2} \cdot \left(\int \|a_n f_n\|_{l^\infty}^p \, d\mu \right)^{1/2}.$$

Since f_n are positive and form an unconditional basic sequence, we have

$$\left\| \sum a_n f_n \right\|_{L^p} \sim \left\| \sum |a_n| f_n \right\|_{L^p} = \| \, \|a_n f_n\|_{l^1} \|_{L^p},$$

so we get

$$\| \, \|a_n f_n\|_{l^1} \|_{L^p} \leqslant C \cdot \| \, \|a_n f_n\|_{l^\infty} \|_{L^p}. \quad \blacksquare$$

Observe that Lemma 3.1 implies

$$\| \, \|a_n f_n\|_{l^1} \|_{L^p} \sim \| \, \|a_n f_n\|_{l^p} \|_{L^p} = \|a_n \|f_n\|_{L^p} \|_{l^p}. \tag{10}$$

We now want to apply this consequence in our situation. We will work with $\mathrm{Re}\, \frac{1}{1-\overline{z_n}z} = \mathrm{Re} \frac{1}{\overline{z_n}} k'_{z_n}$ since these functions are positive. Observe that for a_n real, since $|\frac{1}{\overline{z_n}}| \leqslant C$, we have from (2) that

$$\left\| \sum a_n k'_{z_n} \right\|_{L^q((1-|z|)^{q-2})} \sim \left\| \sum a_n Re \frac{1}{1-\overline{z_n}z} \right\|_{L^q((1-|z|)^{q-2})}. \tag{11}$$

Here we also used the boundedness of the Hilbert transform. So from this and (2) we get that $\mathrm{Re} \frac{1}{1-\overline{z}_n z}$ is an unconditional basic sequence. Applying (10) to $\mathrm{Re}\, \frac{1}{1-\overline{z_n}z}$ and also using (11) we get (3).

In conclusion, we should point out that a simpler proof is available in the Hilbert case $p = 2$. We then have

$$\left\| \sum a_n \frac{k_{z_n}}{\|k_{z_n}\|_{B_2}} \right\|_{B_2}^2 \sim \int_0^1 \left\| \sum r_n(t) a_n \frac{k_{z_n}}{\|k_{z_n}\|_{B_2}} \right\|_{B_2}^2 dt = \|a_n\|_{l^2}^2.$$

In particular, we do not need Lemma 3.1. Also, a different simple proof for $p = 2$ can be found in [7].

(3) implies (4): The boundedness is easy.

$$\begin{aligned}\left\| \left\langle f, \frac{k_{z_n}}{\|k_{z_n}\|_{B_q}} \right\rangle \right\|_{l^p} &= \sup_{\|a_n\|_{l^q}=1} \left| \left\langle f, \sum a_n \frac{k_{z_n}}{\|k_{z_n}\|_{B_q}} \right\rangle \right| \\ &\leqslant \sup_{\|a_n\|_{l^q}=1} \|f\|_{B_p} \cdot \left\| \sum a_n \frac{k_{z_n}}{\|k_{z_n}\|_{B_q}} \right\|_{B_q} \\ &\leqslant C \cdot \|f\|_{B_p}.\end{aligned}$$

To obtain that the map is onto, we apply the Banach theorem (see e.g. [11, 4.13]). Since the adjoint operator sends $\{a_n\}$ to $\sum \overline{a_n} \frac{k_{z_n}}{\|k_{z_n}\|_{B_q}}$, the required inequality is just one direction in relation (3). Note that here we use the equivalence of the B_q norm and the norm considered as an element in the dual of B_p.

(4) implies (5): The Carleson measure condition is simply a reformulation of the boundedness of the map. To get the separation condition, we choose f so that $f(z_n) = 1$ and $f(z_m) = 0$ for $m \neq n$. It now follows from the open mapping theorem and Lemma 2.3 that

$$\|f\|_{B_p} \leqslant C \cdot \frac{|f(z_n)|}{\|k_{z_n}\|_{B_q}} \leqslant C \cdot \|f\|_{B_p} \cdot \frac{\beta(z_n, z_m)^{1/q}}{\beta(z_n, 0)^{1/q}}.$$

To end this section, we show that free interpolation for B_p coincides with interpolation in the sense of (4).

It is clear that (4) implies that $\{z_n\}$ is a sequence of free interpolation. We consider the converse. The trace space $l_{B_p} = \{f(z_n) : f \in B_p\}$ can be identified with $B_p / B_p^{z_n}$ where $B_p^{z_n}$ is the subspace of B_p consisting of those functions satisfying $f(z_n) = 0$. This identification gives a norm on l_{B_p}.

Define the operator $T_{w_n}(a_n) = \{w_n a_n\}$ from l_{B_p} to itself, $w_n \in l^\infty$, which is well defined by assumption. Using the closed graph theorem one checks that T_{w_n} is bounded. For fixed a_n in l_{B_p} define the operator $T_{a_n}(w_n) = \{w_n a_n\}$ from l^∞ to l_{B_p}. Again, the closed graph theorem shows it is bounded. The Banach–Steinhaus theorem then gives

$$\|T_{w_n}\| \leqslant C \|w_n\|_{l^\infty}, \tag{12}$$

where C is independent of $\{w_n\}$. Observe that

$$\left\| \sum c_n k_{z_n} \right\|_{B_q} = \sup_{\|f\|_{B_p} \leqslant 1} \left| \left\langle \sum c_n k_{z_n}, f \right\rangle \right| = \sup_{\|f\|_{B_p} \leqslant 1} \left| \sum c_n f(z_n) \right|$$

for any $\{c_n\}$. From this and (12) it is easy to verify (2) and hence (4) follows.

4. PROOF OF THE SUFFICIENCY OF CONDITION (5)

We now assume (5) holds and prove (1). First, we consider the separation condition in (5). It is easy to transform it into the condition

$$1-\rho(z_n,z_m)^2 \leqslant C\cdot(1-|z_n|)^{\lambda},$$

where λ is the constant in (5). Written like this, we see it coincides with the one used in [7].

As in [7], we will make use of the fact that a finite number of points may be added to an interpolating sequence and it will still be interpolating. This is because the original sequence also is a zero sequence. This observation is used in [7] for getting consequences of the separation condition, and we will also need it in Lemma 4.2 below.

To a point z, we associate a region V_z defined by

$$V_z=\{w : w\in U,\ |w-z^*|\leqslant(1-|z|)^{\beta}\}.$$

Here z^* denotes the radial projection $\frac{z}{|z|}$ of z. Suppose two such regions V_{z_n}, V_{z_m} intersect and that z_n is the point closest to the boundary. Then $(1-|z_n|)\leqslant(1-|z_m|)^{\eta}$ and z_m is outside of V_{z_n}. Here, β (<1) and η (>1) are chosen so that $1<\eta<\frac{2\beta-1}{1-\lambda}$ and $\eta\cdot\beta>1$. These choices ensure the first and second properties, respectively. A proof of these consequences of the separation condition can be found in [7].

We also need to choose α and ρ so that $1>\alpha>\rho>\beta$. These two indices will only be needed in defining the support of the function g in Lemma 4.1 below.

We will need a way of constructing a function living essentially in a region V_w, with good estimates on how it behaves for all points. This is contained in the following.

LEMMA 4.1. *Suppose $s>-1$, then to a point w in the disk, we can find g so that*

$$f(z)=\int_U \frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{1+s}}$$

satisfies $f(w)=1$, and for points in V_w the value is estimated by

$$f(z)=c(\gamma(z))+O\left(\log\left(\frac{1}{1-|w|}\right)^{-1}\right).$$

Here $\gamma=\gamma(z)$ is defined by $|z-w^*|=(1-|w|)^{\gamma}$ and $c(\gamma)$ is 0 for $\gamma<\rho$, $\frac{\gamma-\rho}{\alpha-\rho}$ for $\rho\leqslant\gamma\leqslant\alpha$ and 1 for $\gamma>\alpha$. For points outside of V_w, we have *the bound*

$$|f(z)| \leqslant C \cdot \log\left(\frac{1}{1-|w|}\right)^{-(p-1)}.$$

Further, it holds that

$$\int_U |g(\zeta)|^p (1-|\zeta|)^{p-2} \leqslant C \cdot \log\left(\frac{1}{1-|w|}\right)^{-(p-1)}. \tag{13}$$

Proof. Define g by the relation

$$\frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}w)^{1+s}} = K \cdot \log\left(\frac{1}{1-|w|}\right)^{-1} \cdot |\zeta - w^*|^{-2},$$

when ζ lives in the annulus

$$(1-|w|)^\alpha \leqslant |\zeta - w^*| \leqslant (1-|w|)^\rho$$

and is further restricted to a cone with vertex in w^* and fixed small aperture. For all other ζ, g is taken to be zero. Also, K is chosen so that $f(w) = 1$. Observe that from the definition, we get that

$$|g(\zeta)| \leqslant C \cdot \log\left(\frac{1}{1-|w|}\right)^{-1} \cdot |\zeta - w^*|^{-1}.$$

When estimating $f(z)$, we first treat the case that z is in V_w. We split the support of g into $E_1 = \{\zeta\colon |\zeta - w^*| \leqslant (1-|w|)^\gamma\}$ and $E_2 = \{\zeta : |\zeta - w^*| > (1-|w|)^\gamma\}$ and consider the contributions separately. For E_1, we have

$$\begin{aligned}\left|\int_{E_1} \frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{1+s}}\right| &\leqslant C \cdot (1-|w|)^{-\gamma(1+s)} \cdot \int_{E_1} |g(\zeta)|(1-|\zeta|)^s \\ &\leqslant C \cdot \log\left(\frac{1}{1-|w|}\right)^{-1}.\end{aligned}$$

For E_2 we see, after a calculation, that

$$\int_{E_2} \frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}w)^{1+s}} = \frac{\gamma - \rho}{\alpha - \rho}.$$

Since

$$|(1-\bar{\zeta}w)^{-1-s} - (1-\bar{\zeta}z)^{-1-s}| \leqslant C \cdot |z-w| \cdot (1-|\zeta|)^{-2-s},$$

we have as desired

$$\left|\int_{E_2}\frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}w)^{1+s}}-\int_{E_2}\frac{g(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{1+s}}\right|\leqslant C\cdot|z-w|\cdot\int_{E_2}\frac{|g(\zeta)|}{(1-|\zeta|)^2}$$
$$\leqslant C\cdot\log\left(\frac{1}{1-|w|}\right)^{-1}.$$

For a point z outside of V_w we see that $|1-\bar{\zeta}z|\geqslant C\cdot(1-|w|)^{\beta}$ holds when ζ belongs to the support of g. Thus, we have

$$|f(z)|\leqslant C\cdot\log\left(\frac{1}{1-|w|}\right)^{-1}\cdot(1-|w|)^{(\rho-\beta)(1+s)}\leqslant C\cdot\log\left(\frac{1}{1-|w|}\right)^{-(p-1)}.$$

Finally, estimate (13) is a direct calculation. ∎

Observe that whenever $p\leqslant 2$, the estimate on points outside of V_w would also hold if we took $\rho=\beta$, so we would then need not introduce the index ρ. We also remark that the exact value of $c(\gamma)$ will never enter into the applications. However, we have stated it for definiteness.

It seems necessary to motivate our choice of g here. Referring to (8) and (9), we would like g to have at least the same size properties as f'. Since the Bloch space contains the Besov spaces B_p, it is natural for $|g(z)(1-|z|)|$ to be bounded. We also point out that the functions f constructed in Lemma 4.1 behave similar to the real functions in [7, p. 29], the crucial difference being that the functions in Lemma 4.1 are analytic.

From now on, we will assume when referring to Lemma 4.1 that s is fixed for a particular p and chosen large enough for Lemma 2.4 to apply.

The functions in the previous lemma will now be used as 'building blocks' in an inductive construction of an approximating function.

LEMMA 4.2. *Suppose $w_n\in l^\infty$, we can find a_i so that $f=\sum_i a_i f_i$ approximates it in the sense $\|f(z_n)-w_n\|_{l^\infty}<\delta\cdot\|w_n\|_{l^\infty}$ $(0<\delta<1)$. The coefficients a_i as well as $\|f\|_{H^\infty}$ are bounded by $C\cdot\|w_n\|_{l^\infty}$. Here, f_i is the function in Lemma* 4.1 *corresponding to z_i.*

Proof. For simplicity, we normalise by $\|w_n\|_{l^\infty}=1$. We will inductively construct the coefficients, and we consider the points in sequence, ordered by their distance to the boundary. To a point z_1, we pick out an increasing 'chain' of regions $V_{z_1}\subset V_{z_2}\subset\cdots\subset V_{z_k}$ at each step choosing the smallest region strictly containing all the previous ones. Define the numbers β_i as the $c(\gamma_i)$ in Lemma 4.1, where γ_i is given by $|z_{i-1}-z_i^*|=(1-|z_i|)^{\gamma_i}$. The coefficients $a_2,\dots,a_k$ corresponding to $z_2,\dots,z_k$ are already defined. We assume by induction that $|\sum_3^k\beta_i a_i|\leqslant 1$. The coefficient a_1 corresponding to

z_1 is defined by

$$a_1 = w_1 - \sum_2^k \beta_i a_i.$$

To verify the induction hypothesis, we write

$$\sum_2^k \beta_i a_i = \beta_2 \cdot \left(a_2 + \sum_3^k \beta_i a_i \right) + (1 - \beta_2) \cdot \sum_3^k \beta_i a_i$$
$$= \beta_2 \cdot w_2 + (1 - \beta_2) \cdot \sum_3^k \beta_i a_i,$$

keeping in mind that a_2 is defined in the same way as a_1. This implies $|\sum_2^k \beta_i a_i| \leqslant 1$. From that we get $|a_1| \leqslant 2$. We need to check the asserted properties of f. We fix a point z_1 and keep notation as in the construction above. Write

$$f(z_1) - w_1 = \sum_{i=2}^{k} (a_i f_i(z_1) - \beta_i a_i) + \sum_{z_i \neq z_1, \dots, z_k} f_{z_i}(z_1). \tag{14}$$

We consider the first term. Since $|z_1 - z_{i-1}| \leqslant 2 \cdot (1 - |z_i|)^{\eta \cdot \beta} \leqslant 2 \cdot (1 - |z_i|)$ we see that

$$|f_i(z_1) - \beta_i| \leqslant C \cdot \log\left(\frac{1}{1 - |z_i|}\right)^{-1}.$$

By repeatedly applying the separation condition, we have $(1 - |z_i|) \leqslant (1 - |z_k|)^{\eta^{k-i}}$. So in total there is a bound

$$C \cdot \left(\sum_{j=0}^{k-1} \eta^{-j}\right) \cdot \log\left(\frac{1}{1 - |z_k|}\right)^{-1} \leqslant C \cdot \log\left(\frac{1}{1 - |z_k|}\right)^{-1}. \tag{15}$$

We now look to the second term in (14). It will turn out that the contributions are essentially negligible. We first consider the case that z_1 is not in V_{z_i}. We then have the estimate $|f_{z_i}(z_1)| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-(p-1)}$. Suppose now that z_1 is in V_{z_i}. Observe that $|z_i| < |z_1|$. Since V_{z_i} is not in the chain corresponding to z_1, we can find V_{z_j} in the chain that is not contained in V_{z_i} so that $|z_j| \geqslant |z_i|$. Since $diam(V_{z_j}) \leqslant 2(1 - |z_i|)^{\eta \cdot \beta} \leqslant 2(1 - |z_i|)$, we have $|z_1 - z_i^*| \geqslant C \cdot (1 - |z_i|)^{\beta}$. This gives us the estimate $|f_{z_i}(z_1)| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-(p-1)}$.

Putting together the estimates from the various cases, we have

$$|f(z_1) - w_1| \leqslant C \cdot \left(\sup_{z_j} \log\left(\frac{1}{1-|z_j|}\right)^{-1} + \sum_i \log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \right).$$

The last term is finite, as can be seen from the Carleson measure condition. So, the entire expression can be made smaller than a prescribed $\delta < 1$ by removing finitely many points from the sequence.

Finally, we consider the estimate on H^∞ norm. Suppose first z is contained in some region V_{z_1}, which we assume to be the smallest such region. Observe that the estimates on $f_i(z_1)$ can be applied to get $|f(z)| \leqslant C$. For other z, such an estimate is easy to obtain. ∎

We remark that when $p \leqslant 2$, the Carleson measure condition implies that $\sum_i \log(\frac{1}{1-|z_i|})^{-1}$ converges, so for this range the pointwise error estimates would be simpler.

By repeatedly applying Lemma 4.2 we find a function $f = \sum_i a_i f_i$ with $f(z_i) = w_i$ satisfying $|a_i| \leqslant C \cdot \|w_n\|_{l^\infty}$ and $\|f\|_{H^\infty} \leqslant C \cdot \|w_n\|_{l^\infty}$.

The proof of (1) will be completed by an estimate on the multiplier norm of f. Since each f_i comes from a g_i as in Lemma 4.1, f is given by $g = \sum_i a_i g_i$ in representation (9). We first estimate the Carleson norm of $|g|^p(1-|z|)^{p-2}$. Observe that, as a consequence of the separation condition, the supports of g_i are disjoint. Also, note that if the support of g_i intersects a Carleson box $S(I)$, then z_i is contained in $S(2I)$. With this in mind, we have the following:

$$\begin{aligned}
&\int_{\cup_i S(I_i)} |g(z)|^p(1-|z|)^{p-2} \\
&\quad \leqslant \sum_{z_j \in \cup_i S(2I_i)} |a_j|^p \cdot \int_U |g_j(z)|^p(1-|z|)^{p-2} \\
&\quad \leqslant C \cdot \|w_n\|_{l^\infty}^p \cdot \sum_{z_j \in \cup_i S(2I_i)} \log\left(\frac{1}{1-|z_j|}\right)^{-(p-1)} \\
&\quad \leqslant C \cdot \|w_n\|_{l^\infty}^p \cdot C_p\left(\bigcup_i I_i\right). \qquad (16)
\end{aligned}$$

Here we used (13), the Carleson measure condition in (5) and the doubling property of capacity in Lemma 2.1. So we see that the norm in question is bounded by $C \cdot \|w_n\|_{l^\infty}$. We now apply Lemma 2.4 and remark that, by construction, $\|g(z)(1-|z|)\|_{L^\infty} \leqslant C \cdot \|w_n\|_{l^\infty}$, so the Carleson norm of $|f'|^p(1-|z|)^{p-2}$ is also bounded by $C \cdot \|w_n\|_{l^\infty}$. Using Theorem 2.1, the estimate $\|f\|_{M_{B_p}} \leqslant C \cdot \|w_n\|_{l^\infty}$ is completed by recalling that we have $\|f\|_{H^\infty} \leqslant C \cdot \|w_n\|_{l^\infty}$.

5. CONSTRUCTION OF LINEAR EXTENSION OPERATORS

In this section, we give the construction of the extension operators in Theorems 1.2 and 1.3. The ideas are much the same as above: however, we will need more explicit control during the construction. The essential part is contained in the following lemma.

LEMMA 5.1. *We can find h_{z_n} so that $h_{z_n}(z_n) = 1$, $h_{z_n}(z_m) = 0$ for $m \neq n$ and $h_{z_n} = \sum_i a_{z_i,z_n} f_{z_i}$, where f_{z_i} are as in Lemma* 4.1. *The size of the coefficients of a particular f_{z_i} is estimated by*

$$\sum_n |a_{z_i,z_n}| \leqslant C. \tag{17}$$

Proof. To a particular point z_1, we choose regions $V_{z_1} \subset \cdots \subset V_{z_k}$ and numbers β_i as in the proof of Lemma 4.2. We assume numbers c_{z_j,z_m} are already defined, where $1 < j \leqslant k$ and z_m is any point in the sequence. We define $c_{z_1,z_1} = 1$, $c_{z_1,z_2} = -\beta_2$, and recursively

$$c_{z_1,z_j} = (1-\beta_2) \cdot c_{z_2,z_j}, \qquad 2 < j \leqslant k.$$

All other c_{z_1,z_m} are taken to be zero. Observe that

$$\sum_{j=2}^{k} |c_{z_1,z_j}| \leqslant 1, \tag{18}$$

which follows by induction since

$$\sum_{j=2}^{k} |c_{z_1,z_j}| = \beta_2 + (1-\beta_2) \cdot \sum_{j=3}^{k} |c_{z_2,z_j}|.$$

Define

$$h_{z_n}^* = \sum_i c_{z_i,z_n} f_{z_i}.$$

We take $h_{z_n}^*$ as an initial approximation to h_{z_n}. First, we estimate $h_{z_n}^*(z_n)$. By construction, c_{z_i,z_n} is only non-zero for z_i in V_{z_n}. As in the proof of

Lemma 4.2, we have $|f_{z_i}(z_n)| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-(p-1)}$ for such z_i. So we get

$$\left|\sum_{z_i \neq z_n} c_{z_i,z_n} f_{z_i}(z_n)\right| \leqslant C \cdot \sum_{z_i \in V_{z_n}} \log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \leqslant C \cdot C_p(V_{z_n} \cap T)$$
$$\leqslant C \cdot \log\left(\frac{1}{(1-|z_n|)^{\beta}}\right)^{-(p-1)} \leqslant C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}, \quad (19)$$

where we used the Carleson measure condition in (5). Thus, we have

$$h^*_{z_n}(z_n) = 1 + O\left(\log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}\right). \quad (20)$$

We now consider $h^*_{z_n}(z_1)$, where z_1 is a point containing z_n in the chain, say, $z_n = z_j$ in the previous notation. We first look at the contribution coming from those z_i between z_1 and z_j. Suppose $j = 2$, then by choice of c_{z_1,z_2} we have

$$c_{z_1,z_2} + c_{z_2,z_2}\beta_2 = 0.$$

For the case $j > 2$, it follows by definition of c_{z_1,z_j} that

$$c_{z_1,z_j} + \sum_{i=2}^{j} c_{z_i,z_j}\beta_i = c_{z_2,z_j} + \sum_{i=3}^{j} c_{z_i,z_j}\beta_i.$$

By induction, the expression on the left is 0. Now, recalling from the proof of Lemma 4.2. that $|f_{z_i}(z_1) - \beta_i| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-1}$, we use the separation condition as in (15) to get

$$\left|\sum_{i=1}^{j} c_{z_i,z_j} f_{z_i}(z_1)\right| \leqslant C \cdot \sum_{2}^{j} \log\left(\frac{1}{1-|z_i|}\right)^{-1} \leqslant C \cdot \log\left(\frac{1}{1-|z_j|}\right)^{-1}.$$

Next, we consider the remaining z_i. As in the proof of Lemma 4.2, we have $|f_{z_i}(z_1)| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-(p-1)}$. Similar to (19) we get that

$$\left|\sum_{z_i \neq z_1,\ldots,z_j} c_{z_i,z_n} f_{z_i}(z_1)\right| \leqslant C \cdot \sum_{z_i \in V_{z_n}} \log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \leqslant C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}.$$

So in total, we have

$$|h^*_{z_n}(z_1)| \leqslant C \cdot \left(\log\left(\frac{1}{1-|z_n|}\right)^{-1} + \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}\right). \quad (21)$$

Finally, we consider $h^*_{z_n}(z_1)$ for those z_1 not considered above. Since z_n is not contained in the chain associated with z_1, the same must be true for any point z_i for which the coefficient c_{z_i,z_n} is non-zero. So the estimate $|f_{z_i}(z_1)| \leqslant C \cdot \log(\frac{1}{1-|z_i|})^{-(p-1)}$ is available and we have

$$|h^*_{z_n}(z_1)| = \left|\sum_i c_{z_i,z_n} f_{z_i}(z_1)\right| \leqslant C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}, \tag{22}$$

which follows as in (19).

We now correct $h^*_{z_n}$ using the methods in the previous section. More precisely, using Lemma 4.2 repeatedly gives a function $g_{z_n} = \sum_i b_{z_i,z_n} f_{z_i}$ satisfying $g_{z_n}(z_n) = 1 - h^*_{z_n}(z_n)$ and $g_{z_n}(z_j) = -h^*_{z_n}(z_j)$ $(j \neq n)$. We then put $h_{z_n} = h^*_{z_n} + g_{z_n}$.

Estimates (20)–(22) give bounds on the values g_{z_n} is interpolating. So

$$|b_{z_i,z_n}| \leqslant C \cdot \left(\log\left(\frac{1}{1-|z_n|}\right)^{-1} + \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}\right) \tag{23}$$

holds for any z_i. For z_m not having z_n in its chain we need a better bound. We claim that it holds

$$|b_{z_m,z_n}| \leqslant C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}. \tag{24}$$

We may assume $p > 2$ since otherwise estimates (23) and (24) are the same. To verify (24), we refer to the proof of Lemma 4.2. After one iteration in the construction of g_{z_n} the coefficient corresponding to a point z_m has a bound

$$\delta \cdot C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)} + C \cdot \log\left(\frac{1}{1-|z_n|}\right)^{-1} \cdot C' \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}, \tag{25}$$

where $\delta < 1$. The first term is the total contribution from the points where (22) holds. The second from those points where only (21) holds, these points are in V_{z_n} and a computation similar to (19) verifies the above bound.

Assuming $\log(\frac{1}{1-|z_n|})^{-1}$ is small enough, (25) is bounded by $\gamma \cdot C \cdot \log(\frac{1}{1-|z_n|})^{-(p-1)}$ where $\gamma < 1$. From this it is clear that (23) holds. We now check (17). Let z_1 be a point in the sequence and use notation as in the beginning of the proof. We have by definition that $a_{z_1,z_n} = c_{z_1,z_n} + b_{z_1,z_n}$. Since we already have (18), we look at

$$\sum_n |b_{z_1,z_n}| = \sum_{j=1}^{k} |b_{z_1,z_j}| + \sum_{z_n \neq z_1,\ldots,z_k} |b_{z_1,z_n}|.$$

To see this is bounded, use (23) on the first term on the right and (24) on the second. ∎

Suppose $\{h_{z_n}\}$ are the functions in Lemma 5.1, then the extension operator in Theorem 1.2 is given by

$$\{w_n\} \mapsto \sum_n w_n h_{z_n}.$$

To check it is continuous, we remark that the extension is given by

$$\sum_i g_{z_i} \sum_n w_n a_{z_i, z_n}$$

in representation (9). Condition (17) gives that the inner sum is bounded by $C \cdot \|w_n\|_{l^\infty}$, so the estimate on the multiplier norm goes in the same way as in (16).

Using Lemma 5.1 we also prove Theorem 1.3. Choose $\{w_n\}$ so that $\{\frac{w_n}{\|k_{z_n}\|_{B_q}}\}$ is in l^p and let $\{h_{z_n}\}$ be the functions in Lemma 5.1. We estimate the B_p norm of $\sum_n w_n h_{z_n}$. Recall that $f'_{z_i} = C_s \cdot T_s(g_{z_i})$. So changing the order of summation, we have

$$\left(\sum_n w_n h_{z_n}\right)' = C_s T_s\left(\sum_i g_{z_i} \sum_n w_n a_{z_i, z_n}\right).$$

T_s is bounded on $L^p((1-|z|)^{p-2})$ and the supports of g_i are disjoint, so we need to bound

$$\sum_i \left(\int_U |g_{z_i}|^p (1-|z|)^{p-2} \left|\sum_n w_n a_{z_i, z_n}\right|^p\right). \tag{26}$$

For fixed i we use (13) on the first term and split the second to get

$$\sum_i \left(\log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \left|\sum_{n, V_{z_n} \ni z_i} w_n a_{z_i, z_n}\right|^p\right) \tag{27}$$

$$+ \sum_i \left(\log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \left|\sum_{n, V_{z_n}^c \ni z_i} w_n a_{z_i, z_n}\right|^p\right). \tag{28}$$

One should think of (28) as an error term. We continue with (27). Applying Hölder and (17), we get

$$\left|\sum_n w_n a_{z_i,z_n}\right| \leqslant \left(\sum_n |w_n|^p\right)^{1/p} \left(\sum_n |a_{z_i,z_n}|^q\right)^{1/q}$$
$$\leqslant \left(\sum_n |w_n|^p\right)^{1/p} \sum_n |a_{z_i,z_n}| \leqslant C\left(\sum_n |w_n|^p\right)^{1/p},$$

where summation is over the n so that $V_{z_n} \ni z_i$. This means (27) is bounded by

$$\sum_i \left(\log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \sum_{n, V_{z_n} \ni z_i} |w_n|^p\right).$$

Changing the order of summation and applying (5), we get

$$\sum_n |w_n|^p \sum_{z_i \in V_{z_n}} \log\left(\frac{1}{1-|z_i|}\right)^{-(p-1)} \leqslant C \sum_n |w_n|^p \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}.$$

This completes the estimate of (27). We turn to (28). From (24) and Hölder, we have for any i that

$$\left|\sum_{n, V_{z_n}^c \ni z_i} w_n a_{z_i,z_n}\right|^p \leqslant C \sum_n |w_n|^p \log\left(\frac{1}{1-|z_n|}\right)^{-(p-1)}.$$

Inserting this into (28) gives the required estimate of (28) and hence of (26).

Denote

$$\mu = \sum \|k_{z_n}\|_{B_q}^{-p} \delta_{z_n},$$

so that μ is the measure in condition (5). In the proof of Lemma 5.1 and Theorem 1.3 the Carleson measure condition (5), (6) is only used for boxes and not for collections of boxes. Thus, we have the following.

COROLLARY 5.1. *Suppose that* $\{z_n\}$ *satisfies the separation condition in* (5) *and that* $\mu(S(I)) \leqslant C \cdot \log(\frac{1}{|I|})^{-(p-1)}$ *for all intervals* I. *Then Theorem* 1.3 *still holds.*

Thus, when μ satisfies (6) for intervals, the map $f \mapsto \{\frac{f(z_n)}{\|k_{z_n}\|_{B_q}}\}$ is onto l^p. For this map also to be into l^p we need (6) for collections of intervals.

We now give a proof of Theorem 1.2 using ideas from [15]. The result in [15] is formulated for uniform algebras and cannot be cited directly; however, the method can readily be adapted. The resulting proof is may be more easy than the one given before, but we remark that we have not been able to prove Theorem 1.3 using this method.

Fix an integer N, we first look only at the points $z_1, \ldots, z_N$ in the sequence. Denote

$$v_n(j) = \exp\left(2\pi i n \frac{j}{N}\right), \qquad 0 \leqslant j < N.$$

Choose $f_j \in M_{B_p}$, $\|f_j\|_{M_{B_p}} \leqslant M$, so that

$$f_j(z_k) = v_k(j).$$

The proof of Theorem 1.1 states that

$$f_j(z) = \int_U \frac{\varphi_j(\zeta)(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{1+s}} \tag{29}$$

for some φ_j satisfying $|\varphi_j| \leqslant |\phi|$, where ϕ is independent of j and $|\phi|^p(1-|z|)^{p-2}$ is a Carleson measure for B_p. Define

$$g_n = \frac{1}{N}\sum_{j=1}^{N} f_j \overline{v_n(j)},$$

so that $g_n(z)$ is the nth fourier coefficient of $\{f_j(z)\}$ viewed as a function of j. Observe that

$$g_n(z_k) = \frac{1}{N}\sum_{j=1}^{N} v_k(j)\overline{v_n(j)} = \delta_{n,\,k},$$

since the characters $\{v_n\}$ are orthonormal. Define $h_n = g_n^2$. The property $h_n(z_k) = \delta_{n,k}$ follows from a similar one on g_n. Applying Plancherel, we have

$$\left|\sum_{n=1}^{N} w_n h_n\right| \leqslant \|w_n\|_\infty \left(\sum_{n=1}^{N} |g_n|^2\right) \leqslant C\|w_n\|_\infty. \tag{30}$$

We now check the norm estimate. Using Schwarz inequality

$$\left|\left(\sum_{n=1}^{N} w_n h_n\right)'\right| = 2\left|\sum_{n=1}^{N} w_n g_n g_n'\right| \leqslant 2\|w_n\|_\infty \left(\sum_{n=1}^{N} |g_n|^2\right)^{1/2} \left(\sum_{n=1}^{N} |g_n'|^2\right)^{1/2}.$$

We apply Plancherel on the middle factor to see it is bounded. For the last factor observe that

$$g'_n(z) = \int_U \left(\frac{1}{N} \sum_{j=1}^{N} \varphi_j(\zeta)\overline{v_n(j)} \right) \frac{\bar{\zeta}(1-|\zeta|)^s}{(1-\bar{\zeta}z)^{2+s}}.$$

The expression in parentheses is the nth fourier coefficient of $\{\varphi_j(\zeta)\}$ as a function of j. Minkowski's inequality followed by Plancherel gives

$$\left(\sum_{n=1}^{N} |g'_n|^2 \right)^{1/2} \leqslant \int_U |\phi(\zeta)| \frac{(1-|\zeta|)^s}{|1-\bar{\zeta}z|^{2+s}}. \tag{31}$$

Letting N tend to ∞ and taking weak limits, we get functions h_n, g_n satisfying the same estimates as (30) and (31). Arguing as in Lemma 2.4, we get from (31) that $|(\sum_n w_n h_n)'|^p (1-|z|)^{p-2}$ is a Carleson measure for B_p. Together with (30) and Theorem 2.1 this proves Theorem 1.2.

As a corollary to Theorem 1.2, we have that M_{B_p} contains a closed complemented subspace isomorphic to l^∞. This can be established exactly as for H^∞, see [5, p. 295].

6. REMARKS AND EXAMPLES

While condition (5) together with (6) give a complete characterisation of interpolating sequences, it may be difficult to check (6) in concrete instances, especially since it involves a lower bound on the capacity of an arbitrary collection of intervals. If some assumptions are made on how dispersed the intervals are, then general inequalities of this type exist. Specifically, suppose I_i is a collection of intervals so that the intervals of same centre as I_i and of length $\log(\frac{1}{|I_i|})^{-(p-1)}$ are disjoint. Then a 'quasiadditivity' of capacity holds, in the sense that

$$\sum_i C_p(I_i) \leqslant C \cdot C_p\left(\bigcup_i I_i \right)$$

(see [1, Theorem 5]). We can use this to produce examples of interpolating sequences. Suppose $\{z_n\}$ satisfies the necessary condition

$$\mu(S(I)) \leqslant C \cdot \log\left(\frac{1}{|I|} \right)^{-(p-1)},$$

where $\mu = \sum_i \log(\frac{1}{1-|z_i|})^{-(p-1)} \delta_{z_i}$. Further, consider the intervals of centre z_i^* and length $1-|z_i|$, and suppose that for any two such intervals where none of them is contained in the other, the expanded intervals of the same centre and length $\log(\frac{1}{|I|})^{-(p-1)}$ are disjoint. Then μ is a Carleson measure for B_p.

We conclude with some remarks on possible extensions of the present results. It would be nice to have a proof that (2) implies (1) without using the characterisation (5), as an analogue of the Pick property which is available in the Hilbert case $p = 2$. (see [7]).

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